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THE LAGRANGE MULTIPLIERS ASSOCIATED WITH THE ROTATION MATRIX CHARACTERIZING THE MOTION OF A RIGID BODY ABOUT ITS CENTRE OF MASS†

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To describe the motion of a rigid body, parametrization based on the use of a rotation matrix consisting of nine components is chosen instead of angular parameters. The equations of motion of mechanical systems consisting of many bodies coupled to one another turn out to be linear. The description of the rotations is provided by six Lagrange multipliers, grouped in a symmetrical 3×3 matrix, denoted by Λ, the components of which are related to the volume averages of the internal couplings in the body. The following properties are proved for a rigid body rotating about its centre of mass: the negative of the Lagrange multiplier matrix is positive, and at each instant of time an orthonormalized basis exists in which new components of the matrix Λ are constant, which gives six first integrals of the equations of motion [1]. It is proved that three eigenvalues of the matrix Λ do not change with time and, moreover, they can be found in explicit form.

To obtain the optimal trajectories of systems of many bodies coupled to one another, parametrization of their configuration is used as the first step. After this, the equations of motion are written out and the problem of optimization is formulated, for example, in the form of Pontryagin's maximum principle. The non-linear system of differential equations with mixed (initial and terminal) boundary conditions thus obtained is generally solved by the "shooting method". However, its realization very delicate at the initial stage, since there is no numerical information on the associated variables (multipliers), introduced within the framework of Pontryagin's principle. These multipliers have to be "guessed" [1]. Moreover, their mechanical interpretation, if, of course, it exists, is not always obvious.

In this connection it turns out to be reasonable to eliminate the non-linearity of the equations of dynamics. It was suggested in [2] that one could dispense with parametrization of the rotations using angular variables, which are chosen in a more or less involved way (like, for example, the Euler angles or the Denavit–Hartenberg parameters). Here all nine components of the rotation matrix R (a non-singular 3×3 matrix) are preserved in the equations of motion. However, these nine components are dependent: they are related by six conditions that can be represented in the form of the matrix equation $RR^T = R^TR = I$, which is to be regarded as the constraint equation (I is the identity matrix and the superscript T denotes transposition). This constraint is taken into account by introducing six Lagrange multipliers, grouped in a symmetrical 3×3 matrix Λ.

Using this approach, rotational motion can be modelled as simply as translational one. The classical result consists of the fact that in the equations of translational motion the second derivative with respect to time of the displacement vector of the rigid body considered, multiplied by the mass, already appears in the first term. Within the framework of the proposed approach, a similar property also occurs when describing rotational motion – the second derivative with respect to time of the components of the rotation matrix, multiplied by the inertia matrix, also occurs in the first term of the equations of motion. Here these equations are derived directly both for rotational and for translational motion. The dynamic part of the new equations is linear, which is a considerable advantage from the point of view of numerical investigation.

Below we investigate a number of properties of the matrix of Lagrange multipliers Λ. A simple case of the motion of a rigid body about its centre of mass is investigated, corresponding to the motion of a rigid body about a fixed point, if this point coincides with the centre of mass. Finally, we refine some properties of the matrix Λ in the case of an axisymmetrical body, i.e. in the Euler–Lagrange problem.

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1. THE PRELIMINARY PROPERTIES OF THE MATRIX Λ

The motion of a rigid body about a fixed point is defined by the matrix of its rotations R. In the special case when the fixed point coincides with the centre of mass of the body, the following system was obtained in [1, 2]

$$\ddot{R}K_0 = R\Lambda, \quad R^T R = I \tag{1.1}$$

where K_0 is the constant symmetrical positive-definite Poinsot inertia matrix of the rigid body with respect to its centre of mass, which depends on the geometry and distribution of the mass of the body. It is related to the classical inertia matrix J_0 as follows:

$$J_0 = \operatorname{tr}(K_0)I - K_0 \quad \text{or} \quad K_0 = \operatorname{tr}(J_0)I/2 - J_0 \tag{1.2}$$

The matrices J_0 and K_0 are symmetrical.

Numerical solution of system (1.1) enabled a number of its interesting properties to be established, including the constancy in time and negativity of the eigenvalues of the matrix Λ. We have succeeded in obtaining and proving these properties. Finally, the eigenvalues of the matrix of Lagrange multipliers were obtained in explicit form.

We have the following property.

Property 1. The negative of the matrix of the Lagrange multipliers $(-\Lambda)$ is positive.

Proof. We need to show that the scalar product $\langle -\Lambda V, V\rangle$ is positive for any vector V. It is sufficient to show this solely in the case when V is an eigenvector of the matrix Λ.

By twice differentiating the second relation of system (1.1) and eliminating R from the expression obtained using the first relation of system (1.1), we obtain

$$\Lambda K_0^{-1} + K_0^{-1}\Lambda = -2\dot{R}^T\dot{R} \tag{1.3}$$

In view of the symmetry of the matrices K_0^{-1} and Λ for any vector V the following equality holds

$$\langle V, K_0^{-1}\Lambda V\rangle = -2\langle \dot{R}V, \dot{R}V\rangle \tag{1.4}$$

If V is the eigenvector of the matrix Λ, related to the eigenvalue Λ, the quantity

$$\lambda = -\langle \dot{R}V, \dot{R}V\rangle / \langle V, K_0^{-1}V\rangle \tag{1.5}$$

is negative, since the matrix K_0^{-1}, like the matrix K_0, is positive. Hence, since all the eigenvalues of the matrix Λ are negative, the negative of the matrix Λ is positive.

2. THE CONSTANCY OF THE EIGENVALUES OF THE MATRIX Λ

A numerical check of the negativity of the eigenvalues of the matrix Λ of the Lagrange multipliers showed that these eigenvalues do not change with time (here the correctness of this assertion only concerns the Euler–Lagrange problem).

In order to prove that the eigenvalues of the matrix Λ are constant with time, we only need to verify that the derivatives $\dot{\lambda} = 0$. By virtue of relations (1.1)

$$R^T\ddot{R}K_0 = \Lambda \tag{2.1}$$

and a priori the matrix Λ is not constant. Differentiation of the characteristic equation of the matrix Λ

$$\det(\Lambda - \lambda I) = \det(R^T\ddot{R}K_0 - \lambda I) = 0 \tag{2.2}$$

enables us to obtain certain information on its eigenvalues.

As is well known, differentiation of the determinant of the matrix $A(t)$, which depends on time, gives [3]

$$\frac{d}{dt}\det A = \operatorname{tr}\left[\operatorname{Adj}(A)\frac{dA}{dt}\right] \tag{2.3}$$

where Adj(A) denotes the matrix of the cofactors of the matrix A. Successive differentiation of the characteristic polynomial (2.2) gives a system of equations, from which we obtain

$$\dot{\lambda} = \frac{\operatorname{tr}\left\{[\operatorname{Adj}(R^T\ddot{R}K_0 - \lambda I)]\left[\frac{d}{dt}\left(R^T\ddot{R})K_0\right)\right]\right\}}{\operatorname{tr}[\operatorname{Adj}(R^T\ddot{R}K_0 - \lambda I)]} \tag{2.4}$$

It is not possible to complete the proof of the constancy of the matrix Λ in this way, but numerical experiments which have been carried out enable us to conclude that the numerator of expression (2.4) is equal to zero, while the denominator is non-zero, i.e. $\dot{\lambda} = 0$.

The constancy with time of the eigenvalues of the matrix Λ will be proved below by direct calculation.

3. AN EXPRESSION FOR THE MATRIX Λ IN THE FORM OF A FUNCTION OF THE VECTOR OF THE INSTANTANEOUS ANGULAR VELOCITY

We will denote by $j(u)$ the linear representation which defines a vector product $u \times v$: $j(u)v = u \times v$ corresponding to each vector v. Any skew-symmetric matrix has the form $j(u)$. For example, if the matrix R specifies rotation, its derivative satisfies the relation

$$\dot{R}R^T + R\dot{R}^T = 0$$

or, in other words, the matrix $\dot{R}R^T$ is skew-symmetric. Hence, a vector Ω exists, called the instantaneous angular velocity vector, such that

$$\dot{R} = j(\Omega)R$$

But the matrix $R^T\dot{R}$ is also skew-symmetric, therefore a vector ω exists such that

$$\dot{R} = Rj(\omega) \tag{3.1}$$

The vectors Ω and ω are related by the equation $\Omega = R\omega$. To determine the eigenvalues of the matrix Λ we will use the expression [1]

$$\Lambda = \{[j(\omega)]^2 + j(\dot{\omega})\}K_0 \tag{3.2}$$

It has been shown [1, 2], that Eq. (1.1) is equivalent to the classical law of conservation of angular momentum

$$J_0\dot{\omega} + \omega \times (J_0\omega) = 0 \quad \text{or} \quad \dot{\omega} = J_0^{-1}[(J_0\omega) \times \omega] \tag{3.3}$$

Suppose $u \otimes v$ is the tensor product of the vectors u and v. We will introduce the following notation

$$\xi_0 = \omega \otimes \omega, \quad \xi_{ij} = J_0^i \xi_0 J_0^j, \quad i, j = 0, 1, 2, \ldots$$

$$\eta_0 = \langle \omega, \omega \rangle, \quad \eta_{ij} = J_0^i \eta_0 J_0^j, \quad i, j = 0, 1, 2, \ldots$$

$$\tau_0 = \frac{1}{\det J_0}, \quad \tau_1 = \frac{\operatorname{tr} J_0}{2}, \quad \tau_2 = \frac{(\operatorname{tr} J_0)^2 - \operatorname{tr}(J_0^2)}{2} \tag{3.4}$$

Lemma 1. The law of conservation of angular momentum allows of the equivalent representation

$$j(\dot{\omega}) = \tau_0(\xi_{12} - \xi_{21}) \tag{3.5}$$

Proof. Suppose A is a matrix and u, v and w are three vectors. We will denote the mixed product by (u, v, w). Taking into account the fact that

$$(Au, Av, Aw) = (\det A)(u, v, w)$$

we obtain

$$A^T j(Au)A = (\det A) j(u)$$

Applying this formula to the matrix $A = J_0^{-1}$ and the vector $u = (J_0\omega) \times \omega$ from the second relation of (3.3), we obtain

$$j(\dot{\omega}) = \tau_0^{-1} J_0 j[(J_0\omega) \otimes \omega] J_0 \tag{3.6}$$

after which the formula for the double vector product enables us to convert the expression $j[(J_0\omega) \otimes \omega]$. Using the formula for the double vector product, we have

$$j(v \times w) = w \otimes v - v \otimes w$$

Assuming $v = J_0\omega$ and $w = \omega$, we reduce expression (3.6) to the form (3.5).

Property 2. The matrix Λ of Lagrange multipliers can be expressed as a function of ω only

$$\Lambda = \tau_1(\varepsilon_0 - \eta_0 I) + \eta_{01} - (\xi_{01} + \xi_{10}) + \tau_0 J_0[\xi_{11} - \tau_1(\xi_{01} + \xi_{10}) + \tau_2\xi_0] J_0 \tag{3.7}$$

Proof. Writing the formula for the double vector product in the form

$$j(u)j(v) = v \otimes u - \langle u, v\rangle I$$

and assuming $u = v = \omega$, we have

$$[j(\omega)]^2 = \xi_0 - \eta_0 I$$

Combining this result with the result of Lemma 1 and bearing in mind the second expression of (1.2), the formula for Λ can be represented in the form

$$\Lambda = \tau_1(\xi_0 - \eta_0 I) + \tau_0(\xi_{12} - \xi_{21}) - \xi_{01} + \eta_{01} - \tau_0(\xi_{13} - \xi_{22}) \tag{3.8}$$

whence formula (3.7) follows.

The symmetry of the matrix Λ is not obvious from this expression. But, by virtue of the Hamilton–Cayley theorem, the matrix J_0 satisfies the characteristic equation

$$\tau_0^{-1} I - \tau_2 + 2\tau_1 J_0^2 - J_0^3 = 0$$

whence we can express J_0^3 as a function of J_0 and J_0^2, which enable us to convert (3.8) to the following symmetrical form

$$\Lambda = \tau_1(\xi_0 - \eta_0 I) + \eta_{01} + \tau_0\xi_{22} - \tau_0\tau_1(\xi_{12} + \xi_{21}) - (\xi_{01} - \xi_{10}) + \tau_0\tau_2\xi_{11}$$

4. THE CONSTANCY OF THE EIGENVECTORS OF THE MATRIX Λ

The previous results hold without any special assumptions regarding the form or distribution of the masses of the body. We will now consider the Euler–Lagrange problem. We will assume that the body is symmetrical about a certain axis both as regards it shape and from the point of view of the inertia properties. We will use the following notation: k is the unit vector of this axis, C is the moment of inertia of the rigid body about the axis of symmetry and A is the moment of inertia about an axis passing through the centre of mass and perpendicular to the vector k. In this notation, the inertia tensor of the rigid body of revolution at a fixed point has the form

$$J_0 = AI + (C - A)k \otimes k \tag{4.1}$$

Since $A - C/2$ is the integral over all points of the body of the product of the density of the body and the square of the distance through the plane orthogonal to k and passing through the centre of mass, this quantity is positive.

Note that J_0 is the inertia matrix at the initial instant of time, and the vector k is constant. At the instant t the value of the inertia matrix is

$$J = RJ_0R^T = AI + (C-A)(Rk) \otimes (Rk)$$

By virtue of the fact that the matrix J_0 has a special form, due to the rotational symmetry of the body, not only can J_0^3 be expressed as a function of J_0^2 and J_0, but also J_0^2, and then J_0^3 can also be expressed as a function of J_0, which can be shown using the following result.

Lemma 2. The inertia tensor J_0 possesses the following properties

$$1)\ \tau_1 = A + C/2;\quad 2)\ 1/\tau_0 = A^2C;\quad 3)\ J_0^2 = (C+A)J_0 - ACI;\quad 4)\ \tau_2 = A(A+2C)$$

Proof. The first two properties are obvious. By virtue of the equation $(k \otimes k)^2 = k \otimes k$ we have

$$J_0^2 = A^2I + [(C-A)^2 + 2A(C-A)]k \otimes k$$

whence, from expression (4.1) we have property 3. Property 4 can be derived directly from properties 3 and 1.

Remark. Property 3, which expresses J_0^2 in terms of J_0, is related to the fact that the matrix J_0 has two equal eigenvalues.

Since the matrix J_0 has two different eigenvalues, it is fairly easy to integrate Eqs (3.3). Here, by virtue of results obtained previously [1], the instantaneous angular velocity vector has the form

$$\omega = E\Omega_0,\quad E = \exp[t(C/A-1)\langle k,\Omega_0\rangle j(k)] \tag{4.2}$$

where Ω_0 is the initial value of the instantaneous angular velocity vector Ω.

Since E is the rotation around the vector k, we have the following simple properties

$$EJ_0E^T = J_0,\quad J_0E^T = E^TJ_0,\quad EJ_0 = J_0E \tag{4.3}$$

They follow from the equalities

$$j(k)J_0 = J_0j(k) = Aj(k)$$

which are a consequence of the commutivity of J_0 with all integer powers of $j(k)$ and with the exponent of E.

We will introduce the following notation, similar to notation (3.4),

$$\xi_0^* = \Omega_0 \otimes \Omega_0,\quad \eta_0^* = \langle\Omega_0,\Omega_0\rangle \tag{4.4}$$

$$\xi_{ij}^* = J_0^i\xi_0^*J_0^j,\quad \eta_{ij}^* = J_0^i\eta_0^*J_0^j,\quad i,j = 0,1,2,\ldots$$

To simplify expression (3.8) we will need certain relations which link the quantities (3.4) and (4.6). It is clear from relations (4.3) and (4.4) that

$$\xi_0 = E\xi_0^*E^T,\quad \eta_0 = \eta_0^*,\quad \xi_{ij} = E\xi_{ij}^*E^T,\quad i,j = 0,1 \tag{4.5}$$

We have the following properties.

Property 3. The symmetrical matrix $E^T\Lambda E$ is constant with time, and its form is determined by the right-hand side of (3.7), if, in the latter, we replace the expressions without asterisks by the corresponding expressions with asterisks, i.e. we replace ω by Ω_0. Moreover, the six independent components of the matrix $E^T\Lambda E$ specify the first six integrals of the equations of motion.

Proof. Bearing in mind relations (4.5) and replacing ω by the expression $E\Omega_0$ in relation (3.6) for Λ, we obtain

$$\Lambda = \tau_1(E\xi_0^* E^T - \eta_0^* EE^T) + \eta_0^* EJ_0E^T - E(\eta_{01}^* + \eta_{10}^*)E^T +$$

$$+\tau_0 EJ_0[\xi_{22}^* - \pi_1(\xi_{01}^* + \xi_{10}^*) + \tau_2\Omega_0 \otimes \Omega_0]J_0E^T$$

Since E is a rotation, we obtain the required equality, and hence, the matrix $E^T\Lambda E$ is constant with time.

Theorem. Three eigenvalues of the matrix Λ are constant with time.

Proof. Since E is a rotation, the eigenvalues of the matrix Λ are exactly the same as for the constant matrix $E^T\Lambda E$. Hence, the constancy of the eigenvalues of the matrix Λ can be verified before their explicit calculation.

5. AN EXPLICIT CALCULATION OF THE EIGENVALUES OF THE MATRIX Λ IN THE CASE OF AN AXISYMMETRIC RIGID BODY

We have the following property.

Property 4. The vector $k \times \Omega_0$ is an eigenvector of the matrix $E^T\Lambda E$, corresponding to the eigenvalue $-C/2\langle\Omega_0, \Omega_0\rangle$. Since E is a rotation, this vector is also an eigenvector of the matrix Λ.

Proof. The vector $k \times \Omega_0$ is orthogonal to the two vectors k and Ω_0, so that

$$\xi_0^*(k \times \Omega_0) = 0 \quad (k \otimes k)(k \times \Omega_0) = 0, \quad J_0(k \times \Omega_0) = A(k \times \Omega_0)$$

Applying the linear mapping $E^T\Lambda E$ to the vector $k \times \Omega_0$ we have

$$(E^T\Lambda E)(k \times \Omega_0) = \eta_0^*(-\tau_1 I + J_0)(k \times \Omega_0)$$

The required result follows from the equality $A - \tau_1 = -C/2$.

Remark. Since the matrix $E^T\Lambda E$ is symmetrical, the other two eigenvectors are orthogonal to the vector $k \times \Omega_0$, and hence, they must be sought in the plane defined by the vectors k and Ω_0 or, which is better, by the vector k and the vector $\Omega_0 - \langle k, \Omega_0\rangle k$ orthogonal to it.

Replacing the quantity J_0 in the expression for $E^T\Lambda E$ by expression (4.1) we have

$$E^T\Lambda E = (A + C/2)\xi_0^* + \eta_0^*(J_0 - \pi_1 I) - (\xi_{01}^* + \xi_{10}^*) + A^{-2}C^{-1}\xi_{22}^* -$$

$$-(A + C/2)A^{-2}C^{-1}(\xi_{12}^* + \xi_{21}^*) + (A + 2C)A^{-1}C^{-1}\xi_{11}^*$$

Using relations 1, 2 and 4 from Lemma 2, we can regroup terms and write this relation in the form

$$E^T\Lambda E = (A + 3C/2)\xi_0^* + \eta_0^*[J_0 - (A + C/2)I] - (1 + C/(2A))(\xi_{01}^* + \xi_{10}^*) + A^{-1}\xi_{11}^* \tag{5.1}$$

Suppose

$$\mu = \langle k, \Omega_0\rangle, \quad \zeta_{01} = \Omega_0 \otimes k, \quad \zeta_{01} = k \otimes \Omega_0$$

We will formulate the following result, which is necessary to simplify expression (5.1).

Lemma 3. The following relations hold for the inertia matrix J_0, which has two equal eigenvalues

1) $\xi_{01}^* + \xi_{10}^* = 2A\xi_0^* + (C - A)\mu(\zeta_{01} + \zeta_{10})$;

2) $\xi_{11}^* = A^2\xi_0^* + A(C - A)\mu(\zeta_{01} + \zeta_{10}) + (C - A)^2\mu^2 k \otimes k$

Proof. From Eq. (4.1) we obtain

$$\xi_{11}^* = A(\Omega_0 \otimes \Omega_0) + (C - A)\mu\zeta_{10} \tag{5.2}$$

Property 1 is obtained directly by considering the symmetrical part of (5.2)
Multiplying relation (5.2) on the right by J_0, we obtain the relation

$$\xi_{11}^{*} = A\Omega_0 \otimes (J_0\Omega_0) + (C-A)\mu k \otimes (J_0\Omega_0)$$

from which we obtain Property 2.

The following result can be derived directly from (5.1) using relations 1 and 2 of Lemma 3

$$E^T \Lambda E = -C\eta_0^* I/2 + C\xi_0^*/2 - C/(2A)(C-A)\mu(\zeta_{01}+\zeta_{10}) +$$
$$+(C-A)[\eta_0^* + (C-A)A^{-1}\mu^2]k\otimes k \tag{5.3}$$

If the rigid body is a uniform fall, we have $A = C$, and the matrix Λ can be evaluated directly. We have

$$J_0 = CI, \quad \operatorname{tr} J_0 = 3C, \quad K_0 = CI/2, \quad E = I; \quad \dot{\omega} = 0, \quad \omega = \Omega_0$$

Then

$$\Lambda = C[j(\Omega_0)]^2/2 = C \ \ (\xi_0^* - \eta_0^* I)/2$$

This expression is identical with expression (5.3) if we make the substitution $A = C$. Hence it can be seen that the matrix

$$-\Lambda = Cj(\Omega_0)[j(\Omega_0)]^T/2$$

is positive (but not positive-definite).

Property 5. Suppose $r = [\eta_0^* - \mu^2]^{1/2}$ is the length of the vector $\Omega_0 - \mu k$, and n is the unit vector $r^{-1}(\Omega_0 - \mu k)$. Then the matrix $E^T\Lambda E$ allows of the following tensor representation

$$E^T\Lambda E = -C\eta_0^*(n\times k)\otimes(n\times k)/2 - C\mu^2(n\otimes n)/2 +$$
$$+CA^{-1}(A - C/2)r\mu(n\otimes k + k\otimes n) + (A - C/2)r^2(k\otimes k) \tag{5.4}$$

Proof. From the equality

$$\Omega_0 = rn + \mu k$$

we have sequentially

$$\zeta_{01} + \zeta_{10} = r(n\otimes k + k\otimes n) + 2\mu k\otimes k$$
$$\xi_0^* = r^2(n\otimes n) + r\mu(n\otimes k + k\otimes n) + \mu^2 k\otimes k$$

Substituting the previous diads into (5.3), we obtain

$$E^T\Lambda E = -C\eta_0^* I/2 + Cr^2(n\otimes n)/2 + CA^{-1}(A - C/2)r\mu_0(n\otimes k + k\otimes n) +$$
$$+[(A - C/2)\mu^2 + (C-A)\eta_0^*]k\otimes k$$

But the identity matrix I can be represented in the form

$$I = (n\times k)\otimes(n\times k) + n\otimes n + k\otimes k$$

whence we finally obtain representation (5.4).
We will not obtain the eigenvalues of the matrix Λ. We will introduce the scalars

$$c_1 = -(A - C/2)r^2 - C\mu^2/2, \quad c_2 = -C(A - C/2)(1 - C/A)^2 r^2\mu^2/2$$
$$\Delta = c_1^2 + 4c_2$$

Three eigenvalues of the matrix Λ can be represented as

$$\lambda_1 = \frac{c_1 - \sqrt{\Delta}}{2}, \quad \lambda_2 = \frac{c_1 + \sqrt{\Delta}}{2}, \quad \lambda_3 = -\frac{C}{2}\eta_0^2$$

By virtue of Property 4 the quantity $-C\eta_0^{*}/2$ is an eigenvalue of the matrices $E^T\Lambda E$ and Λ. By relation (5.4) the other two eigenvalues of the matrix are identical with the eigenvalues of the matrix

$$\begin{Vmatrix} -C\mu^2/2 & CA^{-1}(A - C/2)r\mu \\ CA^{-1}(A - C/2)r\mu & -(A - C/2)r^2 \end{Vmatrix} \tag{5.5}$$

The result mentioned holds since the characteristic polynomial of this matrix has the form

$$\lambda^2 - c_1\lambda - c_2 = 0$$

Remarks 1. Since the quantity $(A - C/2)$ is positive, Property 1 can be verified by confining ourselves to proving that the matrix which is the negative of matrix (5.5) is positive.

2. The discriminant Δ of the characteristic polynomial is positive.

We have

$$\Delta = c_1^2 + 4c_2 = (A - C/2)^2 r^4 + (C/2)^2\mu^4 + C(A - C/2)r^2\mu^2[1 - 2(1 - C/A)^2]$$

By (4.2) the quantity

$$1 - 2(1 - C/A)^2 = -1 + 4CA^{-2}(A - C/2)$$

exceeds -1, while the discriminant Δ exceeds the positive quantity

$$[(A - C/2)r^2 - C\mu^2]^2$$

3. It can be verified that three eigenvalues of the matrix Λ are negative. Since the scalar c_1 is negative, it must be shown that the quantity λ_2 is negative. We have the sequence of equivalences

$$\lambda_2 < 0 \Leftrightarrow \sqrt{\Delta} < -c_1 \Leftrightarrow \Delta^2 < c_1^2 \Leftrightarrow c_2 < 0$$

although the negativity of c can be derived from the positivity of $(A - C/2)$.

6. THE CAUCHY CONSTRAINT TENSOR AVERAGED OVER THE VOLUME

The equations of motion of a rigid body (1.1) can be derived from the principle of virtual work using the constraint equations

$$R^T R = I$$

which are taken into account using the matrix of the Lagrange multipliers Λ. On the other hand, the body can be regarded as a continuous medium S. Then, using the principle of virtual work for a continuous medium, one can obtain [1] an expression for the Cauchy constraint tensor σ for the matrix Λ averaged over the volume

$$\int_S \sigma dv = -R\Lambda R^T \tag{6.1}$$

where dv is the element of volume.

The simple expression for the final rotation

$$RE = \exp[tj(\Omega_0 + (C/A - 1)\mu k)]$$

and the constancy of the matrix $E^T\Lambda E$ enable us to represent relation (6.1) in the form

$$-\int_S \sigma dv = RE(E^T \Lambda E)(RE)^T$$

Since RE is a rotation, the integral

$$\int_S (RE)^T \sigma (RE) dv$$

is constant with time.

The theorem and Properties 1 and 3 can be interpreted from the point of view of mechanics as follows: the eigenvalues of the Cauchy constraint tensor, averaged over the volume, are constant with time and positive.

7. CONCLUSION

The approach used in this paper to investigate the problem describes the motion using dependent variables [6]. The choice of the variables is based on parametrization of the motion of a rigid body using the vector of translational displacements and the rotation matrix. The vector of translational displacements is not constrained by any conditions, whereas the matrix is constrained by the condition defining the rotation. This condition is taken into account using the symmetric matrix Λ of Lagrange multipliers. The main advantage of this formulation is the simplicity of the system of equations which arises.

The numerical investigation of the Lagrange multipliers carried out using the proposed approach has enabled us to establish interesting properties of the equations of motion: the eigenvalues of the matrix Λ are negative and do not change with time, and the expressions for them can be represented in the form of explicit formulae. Moreover, it turned out that a simple mechanical interpretation of the Lagrange multiplier matrix exists in the form of a function of the components of the Cauchy constraint tensor. Using the relations obtained between the matrix Λ and the Cauchy tensor, averaged over the volume, it has been suggested that the Tresca elastic limit criterion should be used as the method of optimizing the shape.

A similar approach, based on the use of Λ-type matrices probably enables one to simplify the investigation of the motion of more complex mechanical systems. This simplification may have a considerable advantage because of the linearity of the system.

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